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Fibonacci order in the period-doubling cascade to chaos

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Abstract

In this contribution, we describe how the Fibonacci sequence appears within the Feigenbaum scaling of the period-doubling cascade to chaos. An important consequence of this discovery is that the ratio of successive Fibonacci numbers converges to the golden mean in every period-doubling sequence and therefore the convergence to ϕ, the most irrational number, occurs in concert with the onset of deterministic chaos.
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Two of the most remarkable organizing principles mathematically describing natural and man-made phenomena are the Fibonacci number sequence and the Feigenbaum scaling of the period-doubling cascade to chaos. The Fibonacci sequence describes morphological patterns in a wide variety of living organisms [1], and the asymptotic ratio of successive Fibonacci numbers yields the golden mean. The Feigenbaum scaling [2] for the period-doubling cascade to chaos has been observed in a wide range of dynamical systems, from turbulence to cell biology to chemical oscillators [3,4]. Here we describe how the Feigenbaum scaling and the Fibonacci sequence are intimately intertwined.

Some of the patterns described by the Fibonacci sequence, such as seeds in sunflowers or leaves of artichokes, are now understood in terms of geometric packing models [5]. Other patterns, such as the Archimedean spiral of the nautilus sea shell, remain less well understood, although nonlinear feedback growth models have been advanced in mechanistic descriptions [6]. The characteristic feature of the Fibonacci sequence is that the sum of any two consecutive numbers in the series yields the next number, vis., 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Another characteristic property is that the ratio of any two consecutive numbers in the series asymptotically converges to the universal constant $\phi = 1.618\ldots$, the golden mean.

Diverse physical, chemical, and biological systems that display the period-doubling route to chaos satisfy the scaling relations predicted by the Feigenbaum constants δ and α [3]. The constant δ asymptotically converges to the value $\delta = 4.669\ldots$ and allows predictions of the parameter values for each of the period-doubling bifurcations [4]. For example, knowing δ and the parameter values for two successive bifurcation points, one can predict the parameter values of all of the remaining period-doubling bifurcations. Furthermore, δ allows the prediction of the onset of chaotic behavior, corresponding to the accumulation point of the period-doubling bifurcations.

The constant α reflects the scaling of the dependent variable, where smaller replicas of the system response successively appear with each bifurcation. The value of α asymptotically converges to $2.5029\ldots$, and this value allows prediction of the size of the system response with each bifurcation. For example, the size of the period-2 branch is α times the size of the larger of the two period-4 branches. Moreover, this period-4 branch is α times the size of the smaller of the period-4 branches. Consequently, the size of the period-2 branch is α^2 times the size of the smaller of the period-4 branches. These α scalings are preserved for the higher-order periodic bifurcations throughout the period-doubling cascade.

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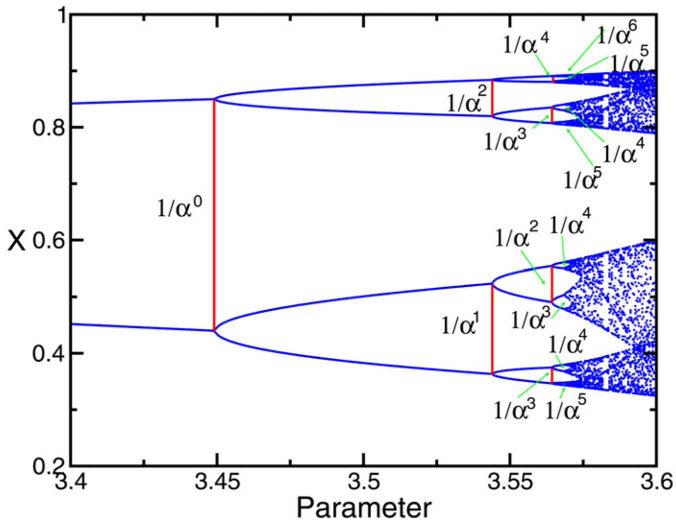

Fig. 1.

Table 1
Distribution of powers of $1/\alpha$ in successive period-doubling bifurcations

	P2 $n=0$	P4 $n=1$	P8 $n=2$	P16 $n=3$	P32 $n=4$	P64 $n=5$		F.N.
$1/\alpha^0$	1							1
$1/\alpha^1$		1						1
$1/\alpha^2$		1	1					2
$1/\alpha^3$			2	1				3
$1/\alpha^4$			1	3	1			5
$1/\alpha^5$				3	4	1		8
$1/\alpha^6$				1	6	5	...	13
$1/\alpha^7$					4	10	...	21
$1/\alpha^8$					1	10	...	34

A typical period-doubling bifurcation diagram for the logistic map is shown in Fig. 1. We normalize the width of the period-2 branch to unity ($1/\alpha^0$), allowing the widths of the branches corresponding to higher periods to be written as inverse powers of α. With this normalization, we notice that the number of branches corresponding to the various powers of $1/\alpha$ follows the sequence:

$$1/\alpha^0,\quad 1/\alpha^1,\quad 2/\alpha^2,\quad 3/\alpha^3,\quad 5/\alpha^4,\quad 8/\alpha^5,\quad 13/\alpha^6,\quad 21/\alpha^7,\quad 34/\alpha^8,\quad 55/\alpha^9,\quad \ldots,$$

where the coefficients $1, 1, 2, 3, 5, 8, \ldots$ correspond to the number of branches with widths $1/\alpha^0$, $1/\alpha^1$, $1/\alpha^2$, $1/\alpha^3$, $1/\alpha^4$, $1/\alpha^5, \ldots,$ respectively. These coefficients are the beginning of the Fibonacci sequence, and we have numerically verified the sequence up to the width $1/\alpha^{21}$, of which there are 17 711 branches. This is consistent with the 22nd term of the Fibonacci sequence. These numerical checks demonstrate the validity of the analytical results (binomial expansion) of Table 1, discussed next, involving the inception of the Fibonacci sequence.

Table 1 reveals how the Fibonacci series develops in the period-doubling cascade. We see that after the first bifurcation gives rise to the period-2 oscillation, there is one segment of width $1/\alpha^0$. An examination of the successive bifurcations shows how the Fibonacci sequence expands with each bifurcation. We also see that there is a pattern of branch widths within each period. Every bifurcation contains a $1/\alpha^n$ branch, which is the first component of a binomial expansion of branch widths according to $(1/\alpha + 1/\alpha^2)^n$ as well as the remaining components of the expansion. The exponent n corresponds to the bifurcation, with $n = 0$ for period-2, $n = 1$ for period-4, and so on.

We emphasize that the Fibonacci sequence will be found in all dynamical systems exhibiting the period-doubling route to chaos, as it is directly linked to the Feigenbaum scaling constant α. However, since the scaling of α holds only in the asymptotic limit, the reported relation also holds only for asymptotic values. It follows that the ratio of successive Fibonacci numbers converges to the golden mean [7], ϕ, in every period-doubling cascade. Thus, the convergence to ϕ, the "most irrational number" [8], occurs in concert with the onset of deterministic chaos.

References

[1] R.V. Jean, D. Barab, Symmetry in Plants, World Scientific, Cambridge, 1998.
[2] M.J. Feigenbaum, J. Stat. Phys. 21 (1979) 669.
[3] H.G. Schuster, Deterministic Chaos: An Introduction, Physik-Verlag, Heidelburg, 1984.
[4] E. Ott, Chaos in Dynamical Systems, Cambridge Univ. Press, Cambridge, 1993.
[5] S. Douady, Y. Couder, J. Theor. Biol. 178 (1996) 255;
S. Douady, Y. Couder, J. Theor. Biol. 178 (1996) 275;
S. Douady, Y. Couder, J. Theor. Biol. 178 (1996) 295.
[6] H. Meinhardt, Models of Biological Pattern Formation, Academic Press, London, 1982.
[7] A.M. Selvan, Appl. Math. Model. 17 (1993) 642.
[8] H. Steinhaus, Mathematical Snapshots, third ed., Dover, New York, 1999, p. 45;
K. Rosen, Elementary Number Theory and its Applications, Addison–Wesley, Boston, 1993.